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AN EXTENSION OF THE TRACE MAP

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Introduction

Let R be a ring, $A\to B$ a finite locally free complete intersection R-morphism. The aim of this article is to extend the usual map $\mathrm{tr}_{B/A}:B\to A$ to a zero degree homogeneous A-linear complex morphism $\Theta_{B/A}:\Omega^\bullet_{B/R}\to\Omega^\bullet_{A/R}$ between the De Rham complexes. This morphism will verify the following properties:

(1) $\Theta_{B/A}$ is a $\Omega^\bullet_{A/R}$-linear morphism (note that $\Theta_{B/A}$ is A-linear).

(2) Let $B\to C$ be another finite locally free R-morphism of complete intersection, then $\Theta_{C/A}=\Theta_{B/A}\circ\Theta_{C/B}$.

(3) There is a compatibility with every change of basis of A.

(4) There is a compatibility with characteristic polynomial in the following way: If $P(X)=\mathrm{Polcar}(X,b)$ is the characteristic polynomial of an element $b\in B$, if $h:C=A[X]/P(X)\to B$ is the morphism obtained by Hamilton-Cayley and if $\mathrm{d}^\bullet h:\Omega^\bullet_{C/R}\to\Omega^\bullet_{B/R}$ is the morphism induced by h, then we have $\Theta_{C/A}=\Theta_{B/A}\circ \mathrm{d}^\bullet h$.

(5) The square

$$\begin{array}{ccc} B^* & \xrightarrow{\mathrm{dlog}_B} & \Omega^\bullet_{B/R} \\ {\scriptstyle N_{B/A}}\downarrow & & \downarrow{\scriptstyle \Theta_{B/A}} \\ A^* & \xrightarrow[\mathrm{dlog}_A]{} & \Omega^\bullet_{A/R} \end{array}$$

commutes. ($N_{B/A}$ is the norm map, dlog is the logarithmic differential: $\mathrm{dlog}\,x=x^{-1}\,\mathrm{d}(x)$).

The construction of such a morphism with properties (1), (2), (3) was known in the case of field extension; its existence had been proved by using residue theory for smooth algebras on a field with characteristic zero (see [4]). When we want to extend the trace morphism in a wider area we have to use completely different methods than the ones precedently used because it is impossible to obtain this morphism by simply generalizing the method used up to now. The hypothesis of complete inter-

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section is essential and the construction is based on the following two properties, first, the relative dualizing module $\mathrm{Hom}_A(B, A)$ is isomorphic to the determinant of the cotangent complex, second, if A is a local henselian ring there exists a deformation $A_0 \to B_0$ of $A \to B$ where A_0 is a henselization of a polynomial ring over $\mathbb{Z}$ and $A_0 \to B_0$ generically etale.

We will adopt the following plan:

(1) Preliminaries on linear algebra.
(2) Construction of Θ.
(3) Independance of the presentation.
(4) Deformation into an etale morphism.
(5) Properties of Θ.
(6) Simple case.
(7) Compatibility with characteristic polynomial and with the norm map.

1. Preliminaries of linear algebra

(In this section, all tensor products and duals are over a ring B, so we don't specify it. Let E be a B-module, we denote by $(\det E)^\vee$ the dual of $\det E$; for any degrees $i \in \mathbb{N}$, $E^i = \bigwedge^i E$ and if $u: E \to F$ is a B-linear map, $u^i = \bigwedge^i u$).

1.1. Lemma. *Let $u: E \to F$ be a B-linear map between projective B-modules of the same rank n. Denote $L = (\det E)^\vee \otimes \det F$; for any degrees i, $1 \le i \le n$, $\tilde{u}^i$ is the composite map*

$$F^i \xrightarrow[\text{can}]{\sim} (F^{n-i})^\vee \otimes \det F \xrightarrow[(u^{n-i})^\vee \otimes 1]{} (E^{n-i})^\vee \otimes \det F \xrightarrow[\text{can}]{} E^i \otimes L.$$

Then we have the following characterisations:

(1) *If we identify $E^i \otimes L$ with $\mathrm{Hom}(E^{n-i}, F^n)$, we have*

$$\forall \eta \in F^i,\ \forall \xi \in E^{n-i},\quad \tilde{u}^i(\eta)(\xi) = \eta \wedge u^{n-i}(\xi).$$

(2) *If E is a free B-module that admits $\{e_1, \ldots, e_n\}$ for a base, denote*

$$\Lambda_n^k = \{\alpha = (\alpha_1, \ldots, \alpha_k), 1 \le \alpha_1 \le \cdots \le \alpha_k \le n, \forall j, \alpha_j \in \mathbb{N}\}$$

and

$$e_\alpha = e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_k} \in E^k$$

and identify L with $\mathrm{Hom}(E^n, F^n)$. Then we have, for any $i \in \mathbb{N}$ and for any $\eta \in F^i$

$$\tilde{u}^i(\eta) = \sum_{\alpha \in \Lambda_n^i} e_\alpha \wedge \lambda_\alpha \quad \text{with } \lambda_\alpha \in L,$$

and such that

$$\forall \beta \in \Lambda_n^{n-i},\quad \sum_{\alpha \in \Lambda_n^i} \lambda_\alpha (e_\alpha \wedge e_\beta) = \eta \wedge u^{n-i}(e_\beta).$$

1.2. Proposition. *Let $u: E \to F$ and $v: E \to G$ be B-linear maps. We suppose that E* *and F are projective B-modules in the same rank n. Denote $Q=\operatorname{coker}(u+v)$ such that we have the exact sequence*

$$E \xrightarrow{u+v} F\oplus G \xrightarrow{q} Q \to 0,$$

$\Delta: B\to L$ the map induced by $\det u: \det E=E^n \to \det f = F^n$ and h the composite map $G \xrightarrow[\mathrm{can}]{} F\oplus G \xrightarrow[q]{} Q$. Then there exists a B-linear map $\underline{\Theta}: \Lambda Q \to \Lambda G\otimes L$ such that the triangle

$$\begin{tikzcd}
\Lambda G \arrow[rr, "\Lambda h"] \arrow[dr, "1_{\Lambda G}\otimes\Delta"'] & & \Lambda Q \arrow[dl, "\underline{\Theta}"] \\
& \Lambda G\otimes L &
\end{tikzcd}$$

commutes.

Proof. For any degrees $i\in\mathbb{N}$ and $j\in\mathbb{N}$, let $\Theta_{i,j}$ be the composite map

$$F^i\otimes G^j \xrightarrow[\tilde{u}^i\otimes 1]{} E^i\otimes L\otimes G^j \xrightarrow[(-1)^i v^i\otimes 1]{} G^i\otimes L\otimes G^j \xrightarrow[\mathrm{can}]{} G^{i+j}\otimes L,$$

$\underline{\Theta}: \Lambda(F\oplus G)\to(\Lambda G)\otimes L$ the sum of $\theta_{i,j}$ and $\tilde{u}^i$ the map defined in Lemma 1.1. Then the diagrams

$$\begin{tikzcd}
E\otimes F^i \arrow[r, "u\otimes 1"] \arrow[d, "1\otimes\tilde{u}^i"'] & F\otimes F^i \arrow[r, "\mathrm{can}"] & F^{i+1} \arrow[d, "\tilde{u}^{i+1}"] \\
E\otimes E^i\otimes L \arrow[rr, "\mathrm{can}\otimes 1"] & & E^{i+1}\otimes 1
\end{tikzcd}$$

commute and when we decompose $\Theta_{i,j}$ we see that $\underline{\Theta}$ vanishes on $\ker(\Lambda q)$ and so induces $\underline{\Theta}: \Lambda Q\to(\Lambda G)\otimes L$ such that $\underline{\underline{\Theta}}=\underline{\Theta}\circ\Lambda q$. Furthermore we have $\tilde{u}^i\otimes u^i = 1_{E^i}\otimes\Delta$. Denote l the canonical injection $l: G\to F\oplus G$. The evaluation of $\underline{\Theta}$ on the factors $F^0\otimes G^j$ shows that we obtain $\underline{\underline{\Theta}}\circ\Lambda l = 1_{\Lambda G}\circ\Delta$ and then we have the property $\underline{\Theta}\circ\Lambda h = 1_{\Lambda G}\circ\Delta$.

1.3. Proposition. *Consider a commutative diagram of the following type:*

$$\begin{tikzcd}
F \arrow[d, "b"'] & E \arrow[l, "u"'] \arrow[r, "v"] \arrow[d, "a"] & G \arrow[d, "c"] \\
F'=F_0\oplus M \arrow[r, "u'"'] & E'=E_0\oplus M \arrow[r, "v'"'] & G'
\end{tikzcd}$$

$$u'=\begin{pmatrix} u_0 & \Phi \\ 0 & 1_M \end{pmatrix}, \qquad v'=(v_0,\psi)$$

in which E, F, E_0, F_0 are projective B-modules of the same rank n, M is a projective B-module of any rank and where a and b are indeed isomorphisms on their image E_0 and F_0 respectively. Denote by

$$\lambda : L = (\det E)^{\vee} \otimes \det F \to L' = (\det E')^{\vee} \otimes \det F'$$

the isomorphism that we deduce, $Q = \operatorname{coker}(u+v)$, $Q' = \operatorname{coker}(u'+v')$, and $\underline{\Theta}$ and $\underline{\Theta}'$ the corresponding maps to $u+v$ and $u'+v'$ given by Proposition 1.2. Then there exists $d : Q \to Q'$ such that we have the following commutative square

$$\begin{tikzcd}
\Lambda Q \arrow[r, "\underline{\Theta}"] \arrow[d, "\Lambda d"'] & \Lambda G \otimes L \arrow[d, "\Lambda c \otimes \lambda"] \\
\Lambda Q' \arrow[r, "\underline{\Theta}'"] & \Lambda G' \otimes L'
\end{tikzcd}$$

Proof. d is the factorisation which is obtained in the diagram

$$\begin{tikzcd}
E \arrow[r, "u+v"] \arrow[d, "a"'] & F \oplus G \arrow[r, "q"] \arrow[d, "b+c"] & Q \arrow[r] \arrow[d, "d"] & 0 \\
E' \arrow[r, "u'+v'"] & F' \oplus G' \arrow[r, "q'"] & Q' \arrow[r] & 0
\end{tikzcd}$$

Now it is sufficient to verify the commutativity of the squares

$$\begin{tikzcd}
F^i \arrow[r, "\tilde{u}^i"] \arrow[d, "b^i"'] & E^i \otimes L \arrow[d, "a^i \otimes \lambda"] \\
F'^i \arrow[r, "\tilde{u}'^i"] & E'^i \otimes L'
\end{tikzcd}$$

For that, first we can suppose $M = 0$, and then we can assume E and M free with basis $\{e_1, \dots, e_n\}$ and $\{e_{n+1}, \dots, e_{n+m}\}$, respectively and use the second characterisation of $\tilde{u}^i$.

1.4. *A few results about complete intersection* (for more details see [2], [4], [5]).

1.4.1. Definition. A finitely generated A-algebra B (or the morphism $A \to B$) is said to be of complete intersection relatively to A if for any ideal $\mathfrak{q}$ in B there exists $t \in B - \mathfrak{q}$ such that the localization B_t is isomorphic to a quotient of a polynomial ring with coefficients in A by a regular sequence.

1.4.2. Let B be a finite locally free complete intersection A-algebra, A' any A-

algebra. Then $B'=A'\otimes_A B$ is a finite locally free complete intersection A'-algebra.

1.4.3. Let $A \to B$ be a finite locally free morphism, let us choose $B \cong C/I$ with $C=A[T_1,\dots,T_n]$ a presentation of B. Then B is of complete intersection if and only if I is locally generated by a regular sequence. Moreover $N_{B/C}=I/I^2$ is a finite locally free B-module.

1.4.4. Let $\omega_{B/A}=(\det N_{B/C})^\vee \otimes_B \det(\Omega^1_{C/A}\otimes_C B)$ be the determinant of the cotangent complex $L^\bullet_{B/A}=(N_{B/C}\to\Omega^1_{C/A}\otimes_C B)$. This determinant is dualizing in the sense that there exists an isomorphism $\omega_{B/A}\xrightarrow{\sim}\mathrm{Hom}_A(B,A)$ and that this determinant is independent, up to isomorphism, of the presentation chosen for B.

Denote $\tau':\mathrm{Hom}_A(B,A)\to A$, $u\mapsto u(1)$; $\tau:\omega_{B/A}\to A$, the map induced by composition, is generally called the 'Tate map' [4].

1.4.5. *Local calculation of* τ. Let $B=A[T_1,\dots,T_n]/(F_1,\dots,F_n)$ be a finite locally free A-algebra so that $F_1,\dots,F_n$ is a regular sequence of $C=A[T_1,\dots,T_n]$. Then $N_{B/C}$ is a free module of rank equal to n. The map $\tau:B\to A$ which gives the duality is characterised in the following way:

Let $p:C\twoheadrightarrow B$ be the canonical surjection, $t=(t_1,\dots,t_n)$ the image of $T=(T_1,\dots,T_n)$ in B^n under the map induced by p (note that $F_i(t)=0$ for $i=1,\dots,n$). Then there exist polynomials $d_{ij}(T)\in B[T_1,\dots,T_n]$ such that

$$F_i(T)=\sum_{j=1}^n d_{i,j}(T)(T_j-t_j)\quad\text{and we have}\quad d_{i,j}(t)=\frac{\partial F_i}{\partial T_j}(t).$$

Set $d(T)=\det(d_{i,j}(T))$; τ is the unique morphism from B to A such that the image of $d(T)$ by the following composite map

$$B\otimes_A C\xrightarrow[\text{can}]{\sim}B[T_1,\dots,T_n]\xrightarrow{\tau\otimes 1}A[T_1,\dots,T_n]\xrightarrow{p}B$$

is equal to 1. (Note that $\mathrm{tr}_{B/A}=d(t)\cdot\tau$ in $\mathrm{Hom}_A(B,A)$.)

More generally, if $\Delta:B\to(\det N_{B/C})^\vee\otimes_C\det\Omega^1_{C/A}=\omega_{B/A}$ is deduced from

$$\delta^\vee:N_{B/C}\to\Omega^1_{C/A}\otimes_C B \text{ by } \det\delta^\vee:\det N_{B/C}\to\det\Omega^1_{C/A}\otimes_C B,$$

then the Tate map $\tau\in\mathrm{Hom}(\Omega_{B/A},A)$ verifies $\mathrm{tr}_{B/A}=\tau\circ\Delta$.

1.4.6. Proposition. *Let $A\to B$ be a finite locally free algebra of complete intersection with $B=C/I$ and $C=A[T_1,\dots,T_n]$. For any A-algebra $A\to A'$, set $B'=A'\otimes_A B$ and $C'=A'\otimes_A B$.*

Then we have a canonial isomorphism

$$(\det N_{B'/C'})^\vee\otimes_{C'}\det(\Omega^1_{C'/A'})\to A'\otimes_A(\det N_{B/C})^\vee\otimes_C\det\Omega^1_{C/A},$$

$\omega_{B'/A}=A'\otimes_A\omega_{B/A}$ is a dualizing module of B' on A' and the Tate map is $\tau'=1_{A'}\otimes\tau$.

2. Construction of Θ

Let R be a ring, let $A\to B$ be a finite locally free R-morphism of complete intersection and let us choose $B=C/I$ with $C=A[T_1,\dots,T_n]$ a presentation of B. When we calculate $\Omega^1_{C/R}$, we see that the exact sequence

$$I/I^2\xrightarrow{\delta}\Omega^1_{C/R}\xrightarrow{\text{can}}\Omega^1_{B/R}\to 0$$

is canonically equivalent to the following one

$$N_{B/C}\xrightarrow{u+v}(\Omega^1_{C/A}\otimes_C B)\oplus(\Omega^1_{A/R}\otimes_A B)\xrightarrow{\omega}\Omega^1_{B/R}\to 0$$

where u and v are the values of δ on the factors $\Omega^1_{C/A}\otimes_C B$ and $\Omega^1_{A/R}\otimes_A B$ respectively. $N_{B/C}$ and $\Omega^1_{C/A}\otimes_C B$ are projective B-modules of the same rank n; using Proposition 1.2 we obtain:

$$\underline{\Theta}:\Lambda_B\Omega^1_{B/R}=\Omega^\bullet_{B/R}\to(\Lambda_B(\Omega^1_{A/R}\otimes_A B))\otimes_B\omega_{B/A}$$

with $\omega_{B/A}=(\det N_{B/C})^\vee\otimes_B(\det(\Omega^1_{C/A}\otimes_C B))$.

The map Θ will be the composite map:

$$\Omega^\bullet_{B/R}\xrightarrow{\underline{\Theta}}(\Lambda_B\Omega^1_{A/R}\otimes_A B)\otimes_B\omega_{B/A}\xrightarrow{\text{can}}\Omega^\bullet_{A/R}\otimes_A\omega_{B/A}\xrightarrow{1\otimes\tau}\Omega^\bullet_{A/R}$$

Remark: $\tau\circ\Delta=\mathrm{tr}_{B/A}$ and for any $i\in\mathbb{N}$ the following triangle commutes:

$$\begin{tikzcd}
\Omega^i_{A/R}\otimes_A B \arrow[rr, "\text{can}"] \arrow[dr, "1\otimes\mathrm{tr}_{B/A}"'] & & \Omega^i_{B/R} \arrow[dl, "\Theta^i"] \\
& \Omega^i_{A/R} &
\end{tikzcd}$$

The Θ we have obtained is an extension of $\mathrm{tr}_{B/A}$ to $\Omega^\bullet_{B/R}$.

3. Independence of the presentation

Let $B\cong C'/I'$, $C'=A[T_1,\dots,T_n]$, $B\cong C''/I''$, $C''=A[U_1,\dots,U_m]$ be two different presentations of B. They canonically induce a new presentation of B by $B\cong C/I$, $C=A[T_1,\dots,T_n,U_1,\dots,U_m]$. Denote by $\Theta_{C'}$, $\Theta_{C''}$ and Θ_C the extensions of $\mathrm{tr}_{B/A}$ defined in Section 2 and associated with the three presentations mentioned above. We now have to verify, for instance, that $\Theta_{C'}=\Theta_C$. Consider the commutative diagram

$$\begin{tikzcd}
N_{B/C'} \arrow[r, "\delta'"] \arrow[d] & (\Omega^1_{C'/A}\otimes_{C'}B)\oplus(\Omega^1_{A/R}\otimes_A B) \arrow[r] \arrow[d] & \Omega^1_{B/R} \arrow[r] \arrow[d, equal] & 0 \\
N_{B/C} \arrow[r, "\delta"] & (\Omega^1_{C/A}\otimes_C B)\oplus(\Omega^1_{A/R}\otimes_A B) \arrow[r] & \Omega^1_{B/R} \arrow[r] & 0
\end{tikzcd}$$

We have a canonical isomorphism $\Omega^1_{C/A}\otimes_C B \xleftarrow{\sim} (\Omega^1_{C'/A}\otimes_{C'} B)\oplus M$ where M is the free B-module of $\Omega^1_{C/A}\otimes_C B$ generated by $\{d_{C/A}(U_j)\otimes 1\}$, $j=1,\dots,m$. We also have a canonical isomorphism $N_{B/C'}\oplus N \xrightarrow{\sim} N_{B/C}$ where N is the free B-submodule of $N_{B/C}$ of rank m.

The elements $u_j\in B$ which are the images of the U_j by the canonical isomorphism $C''/I''\to B$ are also the images of some polynomials $G_j(T)\in C'$.

Let ε_j be the class of $U_j-G_j(T)\in C$ modulo I^2. Then, by using these ε_j, we can verify that the following exact sequence splits and so we obtain the decomposition we wanted. (Observe that $\delta(\varepsilon_j)=d_{C/A}(U_j)\otimes 1$.)

Let σ be the canonical isomorphism

$$\sigma : L'=(\det N_{B/C'})^{\vee}\otimes_B \det(\Omega^1_{C'/A}\otimes_{C'} B) \xrightarrow{\sim} L=(\det N_{B/C})^{\vee}\otimes_B \det(\Omega^1_{C/A}\otimes_i B).$$

τ *and* τ' being the Tate maps, consider the diagram

$$\begin{tikzcd}
 & \Lambda_B(\Omega^1_{A/R}\otimes_A B)\otimes_B L \arrow[r, "\sim"', "\text{can}"'] & \Omega^{\bullet}_{A/R}\otimes_A L \arrow[r, "1\otimes\tau"] & \Omega^{\bullet}_{A/R} \arrow[d, equal] \\
\Omega^{\bullet}_{B/C} \arrow[ur, "\Theta_C"] \arrow[dr, "\Theta_{C'}"'] & & & \\
 & \Lambda_B(\Omega^1_{A/R}\otimes_A B)\otimes_B L' \arrow[r, "\sim"', "\text{can}"'] \arrow[uu, "1\otimes\sigma", "\wr"'] & \Omega^{\bullet}_{A/R}\otimes_A L' \arrow[r, "1\otimes\tau'"] & \Omega^{\bullet}_{A/R}
\end{tikzcd}$$

The left-hand triangle commutes. By using the properties of dualizing module $\omega_{B/A}$ and those of the 'Tate traces' we can conclude that after decomposition we have $\Theta_{C'}=\Theta_C$.

4. Deformation into an etale morphism

4.1. Proposition. *Let A be an henselian local ring, $A\to B$ be a finite locally free complete intersection morphism, $a_1,\dots,a_m$ elements of A. We can find a ring A_0 which is a henselization of a polynomial ring with coefficients in the ring $\mathbb{Z}$, a finite free morphism of complete intersection $A_0\to B_0$, elements $a_{0,1},\dots,a_{0,m}$ of A_0 and a local morphism $\Phi : A_0\to A$ so that the square*

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d, "\Phi"'] & B_0 \arrow[d] \\
A \arrow[r] & B
\end{tikzcd}$$

is cocartesian, $\Phi(a_{0,i})=a_i$, $1\le i\le n$, and furthermore if we denote $K=\mathrm{Frac}(A_0)$, then the morphism $K\to K\otimes_{A_0}B_0$ will be etale.

Proof. We can first assume B is a local ring, quotient of a polynomial ring

$A[T_1, \dots, T_n]$ by n elements $F_1, \dots, F_n$ and $A \to B$ a finite local morphism. Choose an integer $d \geq \sup \deg(F_j)$ and denote $A_1 = \mathbb{Z}[.., X_{\mathbf{i},j}, \dots, Y_k, ..]$ with $k = 1, \dots, m$, $j = 1, \dots, n$ and $\mathbf{i} = (i_1, \dots, i_n) \in \mathbb{N}^n$ so that $\sum_{j=1}^n i_j \leq d$. Note $S_1 = A_1[T_1, \dots, T_n]$, $G_j = \sum_{\mathbf{i}} X_{\mathbf{i},j} T^{\mathbf{i}}$ for $j = 1, \dots, n$, $T^{\mathbf{i}} = T_1^{i_1} \cdots T_n^{i_n}$ and $B_1 = S_1/(G_1, \dots, G_n)$. Then there exists a unique Φ such that $\Phi(X_{\mathbf{i},j})$ is the coefficient of $T^{\mathbf{i}}$, $\Phi(Y_k) = a_k$ and the square

$$\begin{array}{ccc} A_1 & \longrightarrow & B_1 \\ {\scriptstyle \Phi}\downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

is cocartesian.

We consider now A_0 the henselization of A_1 above the closed point of A_1 and B_0 the localisation of $A_0 \otimes_{A_1} B_1$ above the image of the closed point of B. We can then verify that the morphism $A_0 \to B_0$ is a solution to Proposition 4.1 and hence we can easily generalize.

In the same way we have

4.2. Proposition. *Let A be a local henselian ring, $A \xrightarrow{f} B \xrightarrow{g} C$, f and g two finite locally free morphisms of complete intersection, $a_1, \dots, a_m$ some elements of A. It is possible then to get a cocartesian diagram*

$$\begin{array}{ccccc} A_0 & \xrightarrow{f_0} & B_0 & \xrightarrow{g_0} & C_0 \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

in which A_0 is a henselization of a polynomial ring on $\mathbb{Z}$, $a_{0,1}, \dots, a_{0,m}$ elements belonging to A_0 and such that $\Phi(a_{0,j}) = a_j$, for $j = 1, \dots, m$. Furthermore f_0, g_0 are finite locally free complete intersection morphisms and if $K = \mathrm{Frac}(A_0)$, then the morphisms $K \to K \otimes_{A_0} B_0$ and $K \to K \otimes_{A_0} C_0$ will be etale.

5. Properties of Θ

5.1. *Reduction to the case $R = \mathbb{Z}$.* Let $A \to B$ be a R-algebra verifying the hypothesis given in the introduction. Denote by $c : \Omega^1_{B/\mathbb{Z}} \to \Omega^1_{B/R}$ and $d : \Omega^1_{A/\mathbb{Z}} \to \Omega^1_{A/R}$ the surjective maps induced by the canonical map $\mathbb{Z} \to R$. Denote by Θ_R and $\Theta_{\mathbb{Z}}$ the extensions of $\mathrm{tr}_{B/A}$ defined in Section 2 in the case where R and $\mathbb{Z}$ are the base rings.

Then we get the following commutative square

$$\begin{tikzcd} \Omega^\bullet_{B/\mathbb{Z}} \arrow[r, "Ac"] \arrow[d, "\Theta_Z"'] & \Omega^\bullet_{B/R} \arrow[d, "\Theta_R"] \\ \Omega^\bullet_{A/\mathbb{Z}} \arrow[r, "Ad"] & \Omega^\bullet_{A/R} \end{tikzcd}$$

Proof. Use the results given in Section 1.

Now we will consider $R=\mathbb{Z}$ and we will denote $\Omega^\bullet_B=\Omega^\bullet_{B/\mathbb{Z}}$ and $\Omega^\bullet_A=\Omega^\bullet_{A/\mathbb{Z}}$.

5.2. *We have compatibility with base change.*

Let A' be an A-algebra, $A\to B$ be a finite locally free morphism of complete intersection. Then the morphism $A'\to A'\otimes_A B$ which is deduced is also a finite locally free morphism of complete intersection.

Denote by Θ and Θ' the extensions of $\mathrm{tr}_{B/A}$ and $\mathrm{tr}_{B'/A'}$ defined in Section 2. Then we have the following commutative diagram:

$$\begin{tikzcd} \Omega^\bullet_B \arrow[r, "\Theta"] \arrow[d] & \Omega^\bullet_A \arrow[d] \\ A'\otimes_A \Omega^\bullet_B \arrow[r, "1_{A'}\otimes\Theta"] \arrow[d] & A'\otimes_A \Omega^\bullet_A \arrow[d] \\ \Omega^\bullet_{B'} \arrow[r, "\Theta"] & \Omega^\bullet_{A'} \end{tikzcd}$$

Proof. Use the results given in Section 1.

5.3. Transitivity. *Consider $A\xrightarrow{f}B\xrightarrow{g}C$, where f and g are two finite locally free morphisms of complete intersection. Then the maps defined in Section 2 verify $\Theta_{C/A}=\Theta_{B/A}\circ\Theta_{C/B}$.*

Proof. We first check transitivity if the maps f and g are generically etale. Now, if ω is an homomogeneous element of $\Omega^\bullet_C$, we can get a deformation $A_0\to B_0\to C_0$ by Proposition 4.2, so that there exists an element $\omega_0\in\Omega^\bullet_{C_0}$ whose image is ω. We conclude by using 5.2.

5.4. *The extension built in Section 2 commutes with the differentials.*

Proof. On the one hand we reduce to the case where A is a local ring by using the injection

$$\Omega_A^\bullet \to \prod_{\mathfrak{m}\in \operatorname{Spec} A} \Omega_A^\bullet \otimes_A A_{\mathfrak{m}} \cong \prod_{\mathfrak{m}\in \operatorname{Spec} A} \Omega_{A_{\mathfrak{m}}}^\bullet,$$

on the other hand, when A is a local ring, we reduce to the local henselian case by using the injection $\Omega_A^\bullet \rightarrowtail \Omega_{A'}^\bullet$ where A' is a henselization of A.

Assume now A is a local henselian ring and $\omega \in \Omega_B^\bullet$ a homogeneous element. We can find by Proposition 4.1 a deformation $A_0 \to B_0$ of $A \to B$, so that there exists $\omega_0 \in \Omega_{B_0}^\bullet$ whose image is ω. We have then to verify the compatibility with differentials for the deformation $A_0 \to B_0$ given in Proposition 4.1. But in that case, $\Omega_{A_0}^\bullet$ embeds in $\Omega_K^\bullet$, ($K = \operatorname{Frac}(A_0)$), so we just have to verify this compatibility for $K \to K \otimes_{A_0} B_0 = C_0$.

This $K \to C_0$ is an etale morphism, so we can find $K \to K'$, a faithfully flat etale morphism such that $C_0' = K' \otimes_K C_0 \cong K'^m$. The extension of $\operatorname{tr}_{C_0'/K'}$ given in Section 2 is the m-fold sum of the identity map $1_{\Omega_{K'}^\bullet}$ and thus commutes with differentials.

5.5. *The map $\Theta : \Omega_{B/R}^\bullet \to \Omega_{A/R}^\bullet$ that has been constructed is $\Omega_{A/R}^\bullet$-linear for the canonical structure of the $\Omega_{A/R}^\bullet$-module $\Omega_{B/R}^\bullet$.*

Proof. We can check this proposition by induction, using the compatibility of with differentials and the commutativity of the triangle

$$\begin{tikzcd} \Omega_{A/R}^\bullet \otimes_A B \arrow[rr] \arrow[dr, "1\otimes \operatorname{tr}_{B/A}"'] & & \Omega_{B/R}^\bullet \arrow[dl, "\Theta_{B/A}"] \\ & \Omega_{A/R}^\bullet & \end{tikzcd}$$

6. Description in the case of a simple algebra

We calculate now Θ in the case where B is an algebra generated by one element. It is the only case where we have explicite formulae. Let $B = A[T]/(F(T))$ with $F(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0$, $a_i \in A$ for $i = 1, \ldots, n-1$. Denote by t the class of T in B and by I the ideal of $A[T] = C$ generated by $F(T)$. $N_{B/A} = I/I^2$ is a free B-module of rank 1 and generated by the class of $(F(T)) \bmod I^2$; $\Omega_{C/A}^1 \otimes_C B$ is a free B-module with $d_{C/A}(T) \otimes 1$ as generator. With respect to these basis the map $u : N_{B/A} \to \Omega_{C/A}^1 \otimes_C B$ defined in Section 2 is simply the multiplication by $F'(t)$. We have $\det(u) = F'(t)$. Denote by $d_A F(I \otimes t)$ the element $\sum_{i=1}^{n-1} d_A(a_i)t^i \in \Omega_A^1 \otimes_A B$; its image $d_B F(t) = \sum_{i=1}^{n-1} d_A(a_i)t^i \in \Omega_B^1$ verifies $d_B F(t) = -F'(t) d_B(t)$.

The map υ defined in Section 2 is then the multiplication by $d_A F(1 \otimes t)$, ($\upsilon : B \to \Omega_A^1 \otimes_A B$). The dualizing module $\omega_{B/A}$ is canonically isomorphic to B and is in $\operatorname{Hom}_A(B, A)$. The Tate map is defined by

$$\tau(t^{n-1}) = 1, \qquad \tau(t^i) = 0 \quad \text{for } i = 0, \ldots, n-2.$$

We denote $\Theta=(\Theta^i)_{i\geq 0}$ and $\Theta^i:\Omega_B^i\to\Omega_A^i$.

Description of Θ^1. If $\omega\in\Omega_B^1$ is the image of an element $\alpha\in\Omega_A^1\otimes B$ by the canonical map we have $\Theta(\omega)=(1\otimes \mathrm{tr}_{B/A})(\alpha)$.

If $\omega=at^kd_B(t)$ with $a\in A$, then $\Theta(\omega)=-1\otimes\tau(at^kd_AF(1\otimes t))$.

By calculation, we can prove the existence of the following relations:

$$\Theta(t^kd_B(t))=-d_A(a_{n-k-1})-a_{n-1}\Theta(t^{k-1}d_B(t))-\cdots-a_{n-k}\Theta(d_B(t))$$
$$\text{for } k=0,\dots,n-1$$

Furthermore choose

$$F_0(t)=1,$$
$$F_k(t)=t^k+a_{n-1}t^{k-1}+\cdots+a_{n-k}\quad\text{for } k=0,\dots,n-1$$

as a base of B on A. Then we have the property

$$\Theta(F_k(t)d_B(t))=-d_A(a_{n-k-1}).$$

Θ is then completely defined since $\Theta^0=\mathrm{tr}_{B/A}$ and for $i\geq 2$, Θ^i is obtained by the commutativity of the following square

$$\begin{tikzcd}
\Omega_A^{i-1}\times\Omega_B^1 \arrow[r, two heads, "\mathrm{can}"] \arrow[d, "{(1,\Theta^1)}"'] & \Omega_B^i \arrow[d, "\Theta^i"] \\
\Omega^{i-1}\times\Omega_A^1 \arrow[r, two heads, "\mathrm{can}"] & \Omega_A^i
\end{tikzcd}$$

(the canonical arrow on the top being surjective in that case).

Example. $R=\mathbb{F}_p$, $A=\mathbb{F}_p(X)$, $B=A[Y]/(Y^p-X)$, p a prime integer, $F(Y)=Y^p-X$, $\mathrm{tr}_{B/A}=0$. Denote by y the image of Y in B. We have

$$\Theta(y^{p-1}d_B(y))=-1\otimes\tau(y^{p-1}d_AF(1\otimes y))=1\otimes\tau(d_A(X)\otimes y^{p-1})=d_A(X)\neq 0.$$

So we have $\mathrm{tr}_{B/A}=0$ and the extension $\Theta\neq 0$.

7. Compatibility with the characteristic polynomial and with the norm

We can now verify property (4) given in the Introduction and we will use the same notations.

It is easy to verify that $\Theta_{C/A}=\Theta_{B/A}\circ \mathrm{d}^{\bullet}h$ when $\Omega_A^{\bullet}$ is a torsion free $\mathbb{Z}$-module, we use the torsion free property and the commutativity of Θ with the differentials.

We now reduce the general case to the case where $\Omega_A^{\bullet}$ is a torsion free $\mathbb{Z}$-module. Let $b\in B$, consider $A_0\to B_0$ a solution of 4.1 above $A\to B$ and such that there exists $b_0\in B_0$ whose image is b. The construction of the characteristic polynomial is respected by a change of basis. More precisely if we denote by $P_0(X) = \mathrm{Polcar}(X, b_0) \in A_0[X]$ the characteristic polynomial, then $P(X) = \mathrm{Polcar}(X, b) \in A[X]$ corresponds to $1 \otimes P_0(X)$.

Let $h_0 : C_0 = A_0[X]/P_0(X) \to B_0$ be the morphism given by Hamiltonian–Cayley. We have the commutative square:

$$\begin{tikzcd} \Omega^\bullet_{C_0} \arrow[r, "d^\bullet h_0"] \arrow[d, "\mathrm{can}"'] & \Omega^\bullet_{B_0} \arrow[d, "\mathrm{can}"] \\ \Omega^\bullet_C \arrow[r, "d^\bullet h"] & \Omega^\bullet_B \end{tikzcd}$$

$\Omega^\bullet_{A_0}$ is a torsion free module and thus $\Theta_{C_0/A_0} = \Theta_{B_0/A_0} \circ d^\bullet h_0$.

Corollary (Compatibility with the norms). *The square*

$$\begin{tikzcd} B^* \arrow[r, "\mathrm{dlog}_B"] \arrow[d, "N_{B/A}"'] & \Omega^\bullet_B \arrow[d, "\Theta"] \\ A^* \arrow[r, "\mathrm{dlog}_A"] & \Omega^\bullet_A \end{tikzcd}$$

commutes, where $N_{B/A}$ is the norm map and dlog *the logarithmic differential.*

Let $b \in B^*$, $P(X) = \mathrm{Polcar}(X, b) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$, with the above notations. Call $N_C : A[X]/P(X) \to A$ the norm map of C in A and x the image of X in C. We have

$$N_B(b) = N_C(x) = (-1)^n a_0 \in A^*,$$

$$b^{-1} = -a_0^{-1}(b^{n-1} + a_{n-1}b^{n-2} + \cdots + a_1).$$

Setting

$$P_{n-1}(X) = X^{n-1} + a_{n-1}X^{n-2} + \cdots a_1,$$

we have

$$\begin{aligned} \Theta_B \circ \mathrm{dlog}_B(b) &= \Theta_C(-a_0^{-1}P_{n-1}(x)d_C(x)) \\ &= a_0^{-1}d_A(a_0) = (N_b(b))^{-1}d_A(N_B(b)). \end{aligned}$$

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